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# Central Limit Theorem for Branching Brownian Motions in Random Environment

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**Abstract** We introduce a model of branching Brownian motions in time-space random environment associated with the Poisson random measure. We prove that, if the randomness of the environment is moderated by that of the Brownian motion, the population density satisfies a central limit theorem and the growth rate of the population size is the same as its expectation with strictly positive probability. We also characterize the diffusive behavior of our model in terms of the decay rate of the replica overlap. On the other hand, we show that, if the randomness of the environment is strong enough, the growth rate of the population size is strictly less than its expectation almost surely. To do this, we use a connection between our model and the model of Brownian directed polymers in random environment introduced by Comets and Yoshida.

**Keywords** Branching Brownian motion · Random environment · Poisson random measure · Central limit theorem · Phase transition · Brownian directed polymer

## 1 Introduction

We consider a branching Brownian motion in time-space random environment. In particular, we are concerned with the fluctuations in the population density and the population growth rate. In this paper, we reveal their properties under the condition that the randomness of the environment is moderated by that of the Brownian motion. We also show the existence

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of the phase transition in terms of the population growth rate in connection with Brownian directed polymers in random environment introduced in [9].

Smith and Wilkinson [20] introduced a model of branching processes in random environment as a generalization of the classical Galton-Watson process (see also [1]). There the offspring distributions are assumed to be independent and identically distributed random variables indexed by generation. The continuous time counterpart is then introduced by Kaplan [14]. These two models are generalized to branching processes with spatial motions in time-space random environment (for instance, see [4, 17, 23]). In particular, Birkner, Geiger and Kersting [4] studied the long time behavior of the population size for a branching random walk in random environment. Furthermore, Yoshida [23] proved a central limit theorem for the population density in terms of convergence in probability by using the square integrability of an associated martingale. Recently, Nakashima [15] refined this result to be in terms of almost sure convergence.

In this paper, we introduce a new continuous time-space model and show the counterparts of the results which are proved by Yoshida [23]. More precisely, we cope with a branching Brownian motion in time-space random environment associated with the Poisson random measure; the places occupied by Poisson points are suitable for particles to live, and the branching rate of each particle is proportional to the number of Poisson points which influence the particle. Our interest lies in the asymptotic behavior of the model in the situation that the randomness of the Brownian motion moderates that of the environment. In such situation, we show that the population density satisfies a central limit theorem and that the growth rate of the population size is the same as its expectation with strictly positive probability (Theorem 2.1 and Corollary 2.3). We also characterize the diffusive behavior of our model in terms of the decay rate of the replica overlap (Proposition 2.4).

The martingale theory works well for studying asymptotic properties of branching Markov processes (for instance, see [1, 6, 18, 22]) and branching Markov processes in random environment (for instance, see [4, 23]) because the total population size becomes a martingale under the normalization with respect to its expectation. This theory is also applied to our model for the characterization of the correlation among particles. Such correlation exists because Brownian particles may be influenced by common Poisson points, and thus the magnitude of the correlation is proportional to the degree to which pairs of particles meet together. We can characterize this magnitude in quantity (see condition (i) in Theorem 2.1 below) by the expected value of the square of an associated martingale $\overline{M}_t$ which we define by (2.3) below. From this, we find that the square integrability of $\overline{M}_t$ is equivalent to say that the correlation among particles is so weak that the situation is similar to the non-random environment case.

As we discuss in Sect. 4 below, our model is closely related to the model of Brownian directed polymers in random environment introduced by Comets and Yoshida [9]. In fact, if we fix an environment, then the expected population size coincides with the so-called partition function of the latter model as we see in (5.1) below. Furthermore, this relation implies that, if the randomness of the environment is strong enough, the growth rate of the population size is strictly less than its expectation almost surely (Corollary 5.3). Namely, our model has the phase transition in terms of the population growth rate.

## 2 Model and Results

### 2.1 Model

A branching process we consider in this paper is defined by the Brownian motion on $\mathbb{R}^d$ and the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ for $\mathbb{R}_+ = [0, \infty)$. Following [24], we first give some notations for them and then construct branching Brownian motions in time-space random environment. We remark that Savits [17] also constructed branching Markov processes in time-space random environment by applying the results by Ikeda, Nagasawa and Watanabe [11–13], but our construction is more direct and self-contained.

Let $\eta$ denote the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with unit intensity on a probability space $(\mathcal{M}, \mathcal{G}, Q)$. Namely, $\eta$ is a non-negative integer valued random measure such that, $\eta(A_1), \dots, \eta(A_n)$ are mutually independent for disjoint and bounded sets $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ and

$$Q(\eta(A) = k) = e^{-|A|}\frac{|A|^k}{k!} \quad \text{for } A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d),$$

where $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ is the family of all Borel measurable sets on $\mathbb{R}_+ \times \mathbb{R}^d$ and $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^{1+d}$. Let $\{\theta_t\}_{t\geq 0}$ be the time shift operator of the Poisson random measure, that is, $\theta_t\eta = \theta_t\eta(\mathrm{d}s, \mathrm{d}x) = \eta(\{t\} + \mathrm{d}s, \mathrm{d}x)$ identically for any $s, t \geq 0$. The notation $\theta_t\eta$ is often abbreviated to $\eta_t$. We denote by $\{\mathcal{G}_t\}_{t\geq 0}$ the family of the sub-$\sigma$-field of $\mathcal{G}$ defined by

$$\mathcal{G}_t = \sigma(\eta(A \cap ((0, t] \times \mathbb{R}^d)), A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)).$$

Let $\mathbf{M} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{B_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d}, \{\theta_t\}_{t\geq 0})$ be the Brownian motion on $\mathbb{R}^d$, where $\{\theta_t\}_{t\geq 0}$ is the time shift operator of paths, that is, for each path $\omega \in \Omega$, $B_t(\theta_s\omega) = B_{t+s}(\omega)$ identically for any $s, t \geq 0$. Note that we use the same notation $\{\theta_t\}_{t\geq 0}$ as the time shift operators of paths and of the Poisson random measure, respectively. Denote by $V_t$ the tube around the graph $\{(s, B_s)\}_{0<s\leq t}$ defined by

$$V_t = V_t(\omega) = \{(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \mid s \in (0, t],\ x \in U(B_s(\omega))\} \quad \text{for } \omega \in \Omega,$$

where $U(x)$ is a closed ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with unit volume.

Let $\tau$ be a non-negative random variable on $(\Omega, \mathcal{F}, P_x)$, independently of the Brownian motion, of exponential distribution with the mean 1; $P_x(\tau > a) = e^{-a}$ for any $a \geq 0$. Fix a parameter $\alpha > 0$ and set

$$S = S(\omega, \eta) = \inf\{t > 0 \mid \alpha\eta(V_t(\omega)) > \tau(\omega)\} \quad \text{for } (\omega, \eta) \in \Omega \times \mathcal{M}.$$

Then

$$P_x(S(\cdot, \eta) > t) = E_x\big[e^{-\alpha\eta(V_t)}\big].$$

Here we note that, if we fix a path $\omega \in \Omega$, $\{\eta(V_t(\omega))\}_{t\geq 0}$ is a standard Poisson process on the half line. In particular, the jump size of this process is equal to one $Q$-a.s. (for instance, see [16, p. 472, Proposition 1.4]). Let $\{p_n\}_{n=0}^\infty$ be a probability function, that is, $p_n \geq 0$ for any $n \geq 0$ and $\sum_{n=0}^\infty p_n = 1$. In the sequel, we assume $p_0 + p_1 < 1$ to avoid the case where the numbers of particles do not increase for branching Brownian motions which are introduced below. We define

$$m^{(q)} = \sum_{n=0}^\infty n^q p_n \quad \text{for } q \geq 0.$$

We also let $I$ be an $\mathbb{N} \cup \{0\}$-valued random variable on $(\Omega, \mathcal{F}, P_x)$, independently of the Brownian motion and $\tau$, associated with $\{p_n\}_{n=0}^\infty$ so that $P_x(I = n) = p_n$.

We now introduce the index sets. Define

$$K^0 = \{(0)\}, \qquad K^1 = \{(1)\}, \qquad K^n = \{(1, k_2, \ldots, k_n) \mid k_2, \ldots, k_n \in \mathbb{N}\} \quad \text{for } n \geq 2$$

$$\text{and} \quad \mathbf{K} = \sum_{n=0}^{\infty} K^n.$$

In addition, it is useful to set

$$\overline{K^0} = \{(0, 1)\}, \qquad \overline{K^n} = K^{n+1} \quad \text{for } n \geq 1 \quad \text{and} \quad \overline{\mathbf{K}} = \sum_{n=0}^{\infty} \overline{K^n}.$$

If $\mathbf{k} = (1, k_2, \ldots, k_n) \in K^n$ for some $n \geq 1$ and $k \in \mathbb{N}$, then we define $\mathbf{k} \cdot k = (1, k_2, \ldots, k_n, k) \in \overline{K^n}$. By the same way, we define $(0) \cdot 1 = (0, 1) \in \overline{K^0}$.

Let $\{B_t^{\mathbf{k}}\}_{t \geq 0}$ and $\tau^{\mathbf{k}}$, $\mathbf{k} \in \mathbf{K}$, be independent copies of $\{B_t\}_{t \geq 0}$ and $\tau$, respectively. Denote by $V_t^{\mathbf{k}}$ the tube $V_t$ associated with the Brownian motion $\{B_t^{\mathbf{k}}\}_{t \geq 0}$, and by $S^{\mathbf{k}}$ the random variable $S$ with $\tau$ and $V_t$ replaced by $\tau^{\mathbf{k}}$ and $V_t^{\mathbf{k}}$, respectively. In addition, we set $I^{(0)} = 1$ and let $I^{\mathbf{k}}$, $\mathbf{k} \in \mathbf{K} \setminus K^0$, be independent copies of $I$, respectively.

We consider the family of random variables $T^{\mathbf{k}}$ and $\{\mathbf{B}_t^{\mathbf{k}}\}_{t \geq 0}$ indexed by $\mathbf{k} \in \mathbf{K}$ on the measurable space $(\Omega \times \mathcal{M}, \mathcal{F} \otimes \mathcal{G})$ as follows; for each fixed $(\omega, \eta) \in \Omega \times \mathcal{M}$, let $T^{(0)}(\omega, \eta) = 0$ and $\mathbf{B}_t^{(0)}(\omega, \eta) = B_t^{(0)}(\omega)$ identically for any $t \geq 0$. We then define inductively for $\mathbf{k} \cdot k \in \overline{\mathbf{K}}$,

$$T^{\mathbf{k} \cdot k} = T^{\mathbf{k} \cdot k}(\omega, \eta) = \begin{cases} T^{\mathbf{k}}(\omega, \eta) + S^{\mathbf{k} \cdot k}(\theta_{T^{\mathbf{k}}(\omega,\eta)}\omega, \theta_{T^{\mathbf{k}}(\omega,\eta)}\eta), & \text{if } k \leq I^{\mathbf{k}}(\omega), \\ \infty, & \text{if } k \geq I^{\mathbf{k}}(\omega) + 1, \end{cases}$$

and

$$\mathbf{B}_t^{\mathbf{k} \cdot k} = \mathbf{B}_t^{\mathbf{k} \cdot k}(\omega, \eta) = \begin{cases} \mathbf{B}^{\mathbf{k}}_{T^{\mathbf{k}}(\omega,\eta)}(\omega, \eta) + B_t^{\mathbf{k} \cdot k}(\omega) - B^{\mathbf{k} \cdot k}_{T^{\mathbf{k}}(\omega,\eta)}(\omega), \\ \quad \text{for } T^{\mathbf{k}}(\omega, \eta) \leq t < T^{\mathbf{k} \cdot k}(\omega, \eta) \text{ if } k \leq I^{\mathbf{k}}(\omega), \\ \Delta, \quad \text{otherwise}, \end{cases}$$

where $\Delta$ is a cemetery point, $T^{(1)} := T^{(0,1)}$ and $\mathbf{B}_t^{(1)} := \mathbf{B}_t^{(0,1)}$. We use the notations $\mathbf{B}_t^{\mathbf{k}}$ and $T^{\mathbf{k}}$ to denote, respectively, the position and the splitting time of the particle with index $\mathbf{k}$ of a branching Brownian motion. More precisely, we can describe our branching Brownian motion as follows:

- At time 0, the Brownian particle with index 1 starts from $\mathbf{B}_0^{(0)}$.
- The Brownian particle with index $\mathbf{k} \in \mathbf{K} \setminus K^0$ splits into $n$ Brownian particles with probability $p_n$ at site $\mathbf{B}^{\mathbf{k}}_{T^{\mathbf{k}}}$ at time $T^{\mathbf{k}}$.
- These Brownian particles, indexed by $\mathbf{k} \cdot 1, \mathbf{k} \cdot 2, \ldots, \mathbf{k} \cdot n$, respectively, start from $\mathbf{B}^{\mathbf{k}}_{T^{\mathbf{k}}}$ independently.

The definition of the splitting time says that each Brownian particle is apt to split if the associated ball with unit volume catches many Poisson points.

Let us introduce the notion of branching Brownian motions in random environment. We define the probability measures $\{\mathbb{P}_x^{\eta}\}_{x \in \mathbb{R}^d}$ and $\{\mathbb{P}_x\}_{x \in \mathbb{R}^d}$ on $(\Omega \times \mathcal{M}, \mathcal{F} \otimes \mathcal{G})$, respectively, by

$$\mathbb{P}_x^{\eta} = P_x \otimes \delta_{\eta} \quad \text{and} \quad \mathbb{P}_x = \int_{\mathcal{M}} Q(\mathrm{d}\eta) \mathbb{P}_x^{\eta},$$

where $\delta_\eta$ is the Dirac measure at $\eta \in \mathcal{M}$. We call $(\Omega \times \mathcal{M}, \mathcal{F} \otimes \mathcal{G}, \{\{\mathbf{B}_t^{\mathbf{k}}\}_{t\geq 0}\}_{\mathbf{k}\in\mathbf{K}}, \{T^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{K}}, \{\mathbb{P}_x^\eta\}_{x\in\mathbb{R}^d})$ the branching Brownian motion in environment $\eta$ with offspring distribution $\{p_n\}_{n=0}^\infty$, and $(\Omega \times \mathcal{M}, \mathcal{F} \otimes \mathcal{G}, \{\{\mathbf{B}_t^{\mathbf{k}}\}_{t\geq 0}\}_{\mathbf{k}\in\mathbf{K}}, \{T^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{K}}, \{\mathbb{P}_x\}_{x\in\mathbb{R}^d})$ the branching Brownian motion in random environment with offspring distribution $\{p_n\}_{n=0}^\infty$.

We denote by $N_t(A)$ the number of particles on the set $A \in \mathcal{B}(\mathbb{R}^d)$ at time $t$, that is,

$$N_t(A) = \sum_{\mathbf{k}\cdot k\in\overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\mathbf{k}\cdot k} \in A\}}.$$

We can then regard $N_t(\cdot)$ as a configuration measure of particles at time $t$. We denote by $\overline{N}_t$ the total number of particles at time $t$, that is, $\overline{N}_t = N_t(\mathbb{R}^d)$. We also use the notation

$$N_t(f) = \sum_{\mathbf{k}\cdot k\in\overline{\mathbf{K}}} f(\mathbf{B}_t^{\mathbf{k}\cdot k}) \mathbf{1}_{\{T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}, \mathbf{B}_t^{\mathbf{k}\cdot k} \in \mathbb{R}^d\}} \quad \text{for } f \in \mathcal{B}_b(\mathbb{R}^d),$$

where $\mathcal{B}_b(\mathbb{R}^d)$ stands for the set of all bounded Borel measurable functions on $\mathbb{R}^d$.

### 2.2 Results

In this subsection, we state the results in this paper. These results are the continuous model versions of those obtained by Yoshida [23] for branching random walks in random environment. In the sequel, we denote by $P$, $\mathbb{P}^\eta$, $\mathbb{P}$, etc. the quantities $P_x$, $\mathbb{P}_x^\eta$, $\mathbb{P}_x$, etc. for $x = 0$, respectively.

For two independent Brownian motions $(\{B_t^1\}_{t\geq 0}, \{P_x^1\}_{x\in\mathbb{R}^d})$ and $(\{B_t^2\}_{t\geq 0}, \{P_x^2\}_{x\in\mathbb{R}^d})$ on $\mathbb{R}^d$, we let $P_{x,y} = P_x^1 \otimes P_y^2$ and abbreviate $P_{x,x}$ to $P_x$. We then have

$$(\{B_t^2 - B_t^1\}_{t\geq 0}, P_x) \stackrel{\mathrm{d}}{=} (\{B_{2t}\}_{t\geq 0}, P), \tag{2.1}$$

where $\stackrel{d}{=}$ means that the both hand sides have the same law.

We now assume that $m^{(1)}$ is finite. Let us define

$$\beta = \log\{m^{(1)} - e^{-\alpha}(m^{(1)} - 1)\} \quad \text{and} \quad \lambda = \lambda(\beta) := e^\beta - 1. \tag{2.2}$$

Set

$$\overline{M}_t = e^{-\lambda t}\overline{N}_t \quad \text{for } t \geq 0. \tag{2.3}$$

Since $\overline{M}_t$ is a non-negative $\mathbb{P}$-martingale by Lemma 3.2 below, there exists a limit $\lim_{t\to\infty} \overline{M}_t =: \overline{M}_\infty$ $\mathbb{P}$-a.s. Define

$$M_t(\mathrm{d}x) = e^{-\lambda t} N_t(\mathrm{d}x) \quad \text{and} \quad \rho(x) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|x|^2}{2}\right).$$

Let $C_b(\mathbb{R}^d)$ stand for the set of all bounded and continuous functions on $\mathbb{R}^d$. We then obtain

**Theorem 2.1** *Assume*

$$d \geq 3, \qquad m^{(1)} > 1 \quad \textit{and} \quad m^{(2)} < \infty.$$

*Then the following conditions are equivalent to each other*:

(i) $E[\exp(\lambda^2 \int_0^\infty |U(B_t^1) \cap U(B_t^2)| \, \mathrm{d}t)] < \infty$;
(ii) $\lim_{t\to\infty} \overline{M}_t = \overline{M}_\infty$ *in* $L^2(\mathbb{P})$;
(iii) $\lim_{t\to\infty} \int_{\mathbb{R}^d} f(x/\sqrt{t}) M_t(\mathrm{d}x) = \overline{M}_\infty \int_{\mathbb{R}^d} f(x)\rho(x)\, \mathrm{d}x$ *in* $L^2(\mathbb{P})$ *for any* $f \in C_b(\mathbb{R}^d)$.

*Remark 2.2* The expected value in condition (i) expresses the degree to which pairs of Brownian particles meet together, or are effected by common Poisson points. In other words, this value measures the magnitude of the correlation among particles. In particular, condition (i) says that the randomness of the Brownian motion moderates that of the environment. Here we would like to add a remark that the equality

$$E\left[\exp\left(\lambda^2 \int_0^\infty |U(B_t^1) \cap U(B_t^2)| \, \mathrm{d}t\right)\right] = E\left[\exp\left(\frac{\lambda^2}{2} \int_0^\infty |U(0) \cap U(B_t)| \, \mathrm{d}t\right)\right]$$

holds by (2.1). Therefore, from [5, Theorem 5.1] and [21, Theorem 2.4], condition (i) is equivalent to say

$$\inf\left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, \mathrm{d}x \,\middle|\, u \in C_0^\infty(\mathbb{R}^d),\ \frac{\lambda^2}{2} \int_{\mathbb{R}^d} u(x)^2 \, |U(0) \cap U(x)| \, \mathrm{d}x = 1 \right\} > 1,$$

where $C_0^\infty(\mathbb{R}^d)$ denotes the totality of smooth continuous functions with compact support in $\mathbb{R}^d$. Moreover, [9, Proposition 4.2.1] yields that condition (i) holds if

$$\beta \in \left(0, \log\left(1 + \frac{\gamma_d}{2r_d}\right)\right),$$

where $r_d = \beta((d+2)/2)^{1/d}/\sqrt{\pi}$ is the radius of $U(0)$ and $\gamma_d$ is the smallest positive zero of the Bessel function $J_{(d-4)/2}$ defined by

$$J_\nu(\gamma) = \left(\frac{\gamma}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-\gamma^2/4)^k}{k!\gamma(\nu+k+1)} \quad \text{for } \gamma \geq 0 \text{ and } \nu > -1.$$

In contrast with $d \geq 3$, when $d = 1$ or 2, the Brownian motion is recurrent and a pair of particles is apt to meet together as we can see from (2.1). Hence the correlation among particles is so strong that condition (i) does not hold.

Condition (ii) implies $\mathbb{P}(\overline{M}_\infty > 0) > 0$, that is, the growth rate of the population size is the same as its expectation with strictly positive probability. Such situation is similar to the non-random environment case. In fact, for a branching Brownian motion in non-random environment with no extinction, the limit of an associated martingale is strictly positive almost surely (see [1, p. 112, Theorem 2]).

Let $\rho_t(\mathrm{d}x)$ be the population density at time $t$ defined by

$$\rho_t(\mathrm{d}x) = \frac{N_t(\mathrm{d}x)}{\overline{N}_t}.$$

We then get

**Corollary 2.3** (Central limit theorem) *Assume*

$$d \geq 3, \qquad m^{(1)} > 1 \quad \text{and} \quad m^{(2)} < \infty.$$

*If one of the conditions in Theorem 2.1 holds, then*

$$\lim_{t\to\infty}\int_{\mathbb{R}^d} f\left(\frac{x}{\sqrt{t}}\right)\rho_t(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\rho(x)\,\mathrm{d}x \quad \text{in } \mathbb{P}(\cdot\,|\,\overline{M}_\infty > 0)\text{-probability}$$

*for any* $f \in C_b(\mathbb{R}^d)$.

Corollary 2.3 says that the population density $\rho_t(\mathrm{d}x)$ converges to the standard normal distribution under the Brownian scale. We note that S. Watanabe proved an almost sure central limit theorem for branching Brownian motions in non-random environment with no extinction (see [1, p. 245]).

Related to the population density $\rho_t(\mathrm{d}x)$, we let

$$\overline{\rho}_t = \sup_{x\in\mathbb{R}^d}\rho_t(U(x)) \quad \text{and} \quad R_t = \int_{\mathbb{R}^d}\rho_t(U(x))^2\,\mathrm{d}x. \tag{2.4}$$

We can then regard $\overline{\rho}_t$ as the density at the most populated ball with unit volume and $R_t$ as the replica overlap by analogy with the spin glass theory. By the same way as that in [9, Theorem 2.3.2], there exists a constant $c = c(d) \in (0, 1)$ such that $c\overline{\rho}_t^2 \le R_t \le \overline{\rho}_t$ for any $t \ge 0$. Furthermore, we can characterize the diffusive behavior of our model in terms of the decay rate of the replica overlap:

**Proposition 2.4** *Assume*

$$d \ge 3, \qquad m^{(1)} > 1 \quad \text{and} \quad m^{(2)} < \infty.$$

*If one of the conditions in Theorem 2.1 holds, then*

$$R_t = O(t^{-d/2}) \quad \text{in } \mathbb{P}(\cdot\mid\overline{M}_\infty > 0)\text{-probability}.$$

## 3 Moments

To prove the results in the previous section, we calculate the moments of $N_t$. In the sequel, we assume that $m^{(1)}$ is finite. We then have

**Lemma 3.1** *For any* $s, t \ge 0$ *and* $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\mathbb{E}_x^\eta[N_{t+s}(f)\,|\,\mathcal{F}_t\otimes\mathcal{G}_t] = \sum_{\mathbf{k}\cdot k\in\overline{\mathbf{K}}}\mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}\}}E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[e^{\beta\eta_t(V_s)}f(B_s)\right] \quad Q\text{-a.s.} \tag{3.1}$$

*In particular,*

$$\mathbb{E}_x^\eta[N_t(f)] = E_x\left[e^{\beta\eta(V_t)}f(B_t)\right] \quad Q\text{-a.s.} \tag{3.2}$$

*Proof* We prove this lemma only for $f \equiv 1$ and $p_2 = 1$ because the proof is done for the general case by a modification of the notation. We fix $t > 0$ and assume $\eta(V_t) = \eta(V_{t-})$ because this equality holds $Q$-a.s. Let $\mu = e^\alpha - 1$. We first show

$$\begin{aligned}\mathbb{P}_x^\eta(T^{\mathbf{k}} \le t) &= P_x(T^{\mathbf{k}}(\cdot,\eta) \le t)\\ &= -\mu^{n-1}E_x\left[\int_{(0,t]}\binom{\eta(V_{s-})}{n-1}\mathbf{1}_{\{n-1\le\eta(V_{s-})\}}\,\mathrm{d}e^{-\alpha\eta(V_s)}\right] \quad \text{for } n \ge 1 \text{ and } \mathbf{k}\in K^n\end{aligned}$$

by induction, where by $\mathrm{d}e^{-\alpha\eta(V_t)}$ we mean the Stieltjes integral associated with $e^{-\alpha\eta(V_t)}$. This equality holds for $n=1$ by definition. Assume that the equality holds for some $n \geq 2$. Then for $\mathbf{k}\cdot k \in \overline{K^n}$ we have

$$\begin{aligned}
&\mathbb{P}_x^\eta(T^{\mathbf{k}\cdot k} \leq t) \\
&\quad = \mathbb{P}_x^\eta(T^{\mathbf{k}} \leq t, T^{\mathbf{k}} + S^{\mathbf{k}\cdot k} \circ \theta_{T^{\mathbf{k}}} \leq t) \\
&\quad = \mu^{n-1} E_x\left[\int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)} \binom{\eta(V_{s-})}{n-1} \mathbf{1}_{\{n-1\leq \eta(V_{s-})\}} E_{B_s}\left[\int_{(0,t-s]} \mathrm{d}e^{-\alpha\eta_s(V_u)}\right]\right] \quad (3.3)
\end{aligned}$$

by the Markov property. Since we know

$$\mathrm{d}e^{-\alpha\eta(V_t)} = -\mu e^{-\alpha\eta(V_t)}\,\mathrm{d}\eta(V_t) = -(1-e^{-\alpha})e^{-\alpha\eta(V_{t-})}\mathrm{d}\eta(V_t) \quad (3.4)$$

and $\eta_s(V_u \circ \theta_s) = \eta(V_{u+s}) - \eta(V_s)$, the last term of (3.3) above is equal to

$$\begin{aligned}
&\mu^{n+1} E_x\left[\int_{(0,t]} \mathrm{d}\eta(V_s) \binom{\eta(V_{s-})}{n-1} \mathbf{1}_{\{n-1\leq \eta(V_{s-})\}} \int_{(s,t]} e^{-\alpha\eta(V_u)}\,\mathrm{d}\eta(V_u)\right] \\
&\quad = \mu^{n+1} E_x\left[\int_{(0,t]} \mathrm{d}\eta(V_u) e^{-\alpha\eta(V_u)} \int_{(0,u)} \binom{\eta(V_{s-})}{n-1} \mathbf{1}_{\{n-1\leq \eta(V_{s-})\}}\,\mathrm{d}\eta(V_s)\right]
\end{aligned}$$

by Fubini's theorem. Noting that

$$\int_{(0,u)} \binom{\eta(V_{s-})}{n-1} \mathbf{1}_{\{n-1\leq \eta(V_{s-})\}}\,\mathrm{d}\eta(V_s) = \sum_{k=n-1}^{\eta(V_{u-})-1} \binom{k}{n-1} \mathbf{1}_{\{n\leq \eta(V_{u-})\}} = \binom{\eta(V_{u-})}{n} \mathbf{1}_{\{n\leq \eta(V_{u-})\}}$$

holds for any $u > 0$ such that $\eta(V_u) \neq \eta(V_{u-})$, we complete the induction. Hence we get

$$\begin{aligned}
\mathbb{E}^\eta[\overline{N}_t] &= \sum_{n=0}^{\infty} \sum_{\mathbf{k}\cdot k \in \overline{K^n}} \mathbb{P}_x^\eta(T^{\mathbf{k}} \leq t < T^{\mathbf{k}\cdot k}) = \sum_{n=0}^{\infty} \sum_{\mathbf{k}\cdot k \in \overline{K^n}} \{\mathbb{P}_x^\eta(T^{\mathbf{k}} \leq t) - \mathbb{P}^\eta(T^{\mathbf{k}\cdot k} \leq t)\} \\
&= 1 - E_x\left[\int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)} (1+2\mu)^{\eta(V_{s-})}\right].
\end{aligned}$$

Then (3.4) implies

$$1 - \int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)}(1+2\mu)^{\eta(V_{s-})} = 1 + (1-e^{-\alpha}) \sum_{k=1}^{\eta(V_t)} (2-e^{-\alpha})^{k-1} = (2-e^{-\alpha})^{\eta(V_t)},$$

which leads to (3.2).

We finally show (3.1). For any $A \in \mathcal{F}_t \otimes \mathcal{G}_t$, we obtain

$$\mathbb{E}_x^\eta\left[\overline{N}_{t+s}; A\right] = \sum_{\mathbf{k}\cdot k \in \overline{\mathbf{K}}} \mathbb{P}_x^\eta(T^{\mathbf{k}} \leq t+s < T^{\mathbf{k}\cdot k}; A).$$

Define for $\mathbf{k} \in K^n$,

$$\mathbf{k}_l = \begin{cases} (k_1, k_2, \ldots, k_l), & \text{if } n \geq 1 \text{ and } 1 \leq l \leq n, \\ (0), & \text{if } n = 0 \text{ and } l = 0. \end{cases}$$

Then by the Markov property, we have

$$\begin{aligned}
&\mathbb{P}_x^\eta(T^{\mathbf{k}} \le t+s < T^{\mathbf{k}\cdot k}, A) \\
&= \sum_{l=0}^{n} \mathbb{P}_x^\eta(T^{\mathbf{k}_l} \le t < T^{\mathbf{k}_l\cdot k_{l+1}}, T^{\mathbf{k}} \le t+s < T^{\mathbf{k}\cdot k}, A) \\
&= -\mathbb{E}_x^\eta\left[E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[\int_{(0,s]} \mathrm{d}e^{-\alpha\eta_t(V_u)}\right]; T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}, A\right] \\
&\quad - \sum_{l=0}^{n-1} \mathbb{E}_x^\eta\left[E_{\mathbf{B}_t^{\mathbf{k}_l\cdot k_{l+1}}}\left[\int_{(0,s]} \mathrm{d}e^{-\alpha\eta_t(V_u)}\left(\mu^{n-l-1}\binom{\eta_t(V_{u-})}{n-l-1}\mathbf{1}_{\{n-l-1\le\eta_t(V_{u-})\}}\right.\right.\right. \\
&\quad \left.\left.\left. - \mu^{n-l}\binom{\eta_t(V_{u-})}{n-l}\mathbf{1}_{\{n-l\le\eta_t(V_{u-})\}}\right)\right]; T^{\mathbf{k}_l} \le t < T^{\mathbf{k}_l\cdot k_{l+1}}, A\right] \\
&=: (\mathrm{I})_{\mathbf{k}\cdot k} - \sum_{l=0}^{n-1} (\mathrm{II})_{\mathbf{k}_l\cdot k_{l+1}} \quad \text{for any } n \ge 1 \text{ and } \mathbf{k}\cdot k \in \overline{K^n}.
\end{aligned}$$

This implies

$$\begin{aligned}
\mathbb{E}_x^\eta\left[\overline{N}_{t+s}; A\right] &= (\mathrm{I})_1 + \sum_{n=1}^{\infty} \sum_{\mathbf{k}\cdot k\in\overline{K^n}} \left((\mathrm{I})_{\mathbf{k}\cdot k} - \sum_{l=0}^{n-1} (\mathrm{II})_{\mathbf{k}_l\cdot k_l}\right) \\
&= \sum_{n=0}^{\infty} \sum_{\mathbf{k}\cdot k\in\overline{K^n}} (\mathrm{I})_{\mathbf{k}\cdot k} - \sum_{n=1}^{\infty}\sum_{l=0}^{n-1} \sum_{\mathbf{k}\cdot k\in\overline{K^l}} 2^{n-l}(\mathrm{II})_{\mathbf{k}\cdot k} \\
&= \sum_{n=0}^{\infty} \sum_{\mathbf{k}\cdot k\in\overline{K^n}} (\mathrm{I})_{\mathbf{k}\cdot k} - \sum_{l=0}^{\infty} \sum_{\mathbf{k}\cdot k\in\overline{K^l}} \sum_{n=l+1}^{\infty} 2^{n-l}(\mathrm{II})_{\mathbf{k}\cdot k}.
\end{aligned}$$

By the same calculation as that in the proof of (3.2), the last term above is equal to

$$\sum_{\mathbf{k}\cdot k\in\overline{\mathbf{K}}} \mathbb{E}_x^\eta\left[E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[e^{\beta\eta_t(V_s)}\right]; T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}, A\right],$$

whence (3.1) follows. □

By the definition of the Poisson random measure, we obtain

$$Q\left[e^{\beta\eta(A)}\right] = e^{\lambda|A|} \quad \text{for } A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d), \tag{3.5}$$

which implies the following:

**Lemma 3.2** *For any $s, t \ge 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$,*

$$\mathbb{E}_x[N_{t+s}(f) \,|\, \mathcal{F}_t \otimes \mathcal{G}_t] = e^{\lambda s} \sum_{\mathbf{k}\cdot k\in\overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}[f(B_s)].$$

*In particular, $\overline{M}_t$ is a martingale on $(\Omega \times \mathcal{M}, \mathcal{F} \otimes \mathcal{G}, \{\mathcal{F}_t \otimes \mathcal{G}_t\}_{t\ge 0}, \mathbb{P}_x)$ and*

$$\mathbb{E}_x[N_t(f)] = e^{\lambda t} E_x[f(B_t)].$$

*Proof* By the same reason as that in the proof of Lemma 3.1, we prove this lemma only for $f \equiv 1$. Since $U(x)$ has a unit volume, it follows from Fubini's theorem, Lemma 3.1 and (3.5) that

$$\begin{aligned}\mathbb{E}_x\left[\overline{N}_{t+s} \mid \mathcal{F}_t \otimes \mathcal{G}_t\right] &= \sum_{\mathbf{k}\cdot k \in \overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} Q\left[E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[e^{\beta\eta_t(V_s)}\right]\right] \\ &= \sum_{\mathbf{k}\cdot k \in \overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} E_{\mathbf{B}_t^{\mathbf{k}\cdot k}}\left[Q\left[e^{\beta\eta_t(V_s)}\right]\right] = e^{\lambda s}\overline{N}_t\end{aligned}$$

for any $s, t \ge 0$, which completes the proof. □

In the sequel, we further assume that $m^{(2)}$ is finite. Let us define

$$c = m^{(2)} - m^{(1)} = \sum_{n=0}^{\infty} n(n-1)p_n \quad \text{and} \quad \tilde{\mu} = e^{-\alpha}\mu = 1 - e^{-\alpha}.$$

We then have

**Lemma 3.3** *For any $f, g \in \mathcal{B}_b(\mathbb{R}^d)$,*

$$\begin{aligned}\mathbb{E}_x^{\eta}[N_t(f)N_t(g)] &= E_x\left[e^{\beta\eta(V_t)} f(B_t)g(B_t)\right] \\ &\quad + c\tilde{\mu} E_x\left[\int_{(0,t]} e^{\beta\eta(V_{s-})} E_{B_s}\left[e^{\beta\eta_s(V_{t-s})} f(B_{t-s})\right]\right. \\ &\quad \left.\times E_{B_s}\left[e^{\beta\eta_s(V_{t-s})} g(B_{t-s})\right] \mathrm{d}\eta(V_s)\right] \quad Q\text{-a.s.} \end{aligned} \tag{3.6}$$

*Proof* We show this lemma for $f \equiv 1$, $g \equiv 1$ and $p_2 = 1$ by the same reason as that in Lemma 3.1. We fix $t > 0$ and assume $\eta(V_t) = \eta(V_{t-})$. A direct calculation then implies

$$\begin{aligned}\overline{N}_t^2 &= \sum_{\mathbf{k}\cdot k \in \overline{\mathbf{K}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} + \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \overline{\mathbf{K}}, \\ \mathbf{k}\cdot k \ne \tilde{\mathbf{k}}\cdot\tilde{k}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} \mathbf{1}_{\{T^{\tilde{\mathbf{k}}} \le t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}\}} \\ &= \overline{N}_t + \sum_{\substack{\mathbf{k}\cdot k, \mathbf{k}\cdot\tilde{k} \in \overline{\mathbf{K}}, \\ k \ne \tilde{k}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot\tilde{k}}\}} + \sum_{\substack{\mathbf{k}\cdot k, \tilde{\mathbf{k}}\cdot\tilde{k} \in \overline{\mathbf{K}}, \\ \mathbf{k} \ne \tilde{\mathbf{k}}}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot k}\}} \mathbf{1}_{\{T^{\tilde{\mathbf{k}}} \le t < T^{\tilde{\mathbf{k}}\cdot\tilde{k}}\}} \\ &=: \overline{N}_t + \text{(I)} + \text{(II)}.\end{aligned}$$

Since

$$\text{(I)} = 2\sum_{\mathbf{k} \in \mathbf{K} \setminus K^0} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot 1}\}} \mathbf{1}_{\{T^{\mathbf{k}} \le t < T^{\mathbf{k}\cdot 2}\}},$$

it follows that

$$\begin{aligned}\mathbb{E}_x^{\eta}[\text{(I)}] &= 2\sum_{n=1}^{\infty}\sum_{\mathbf{k} \in K^n} P_x(T^{\mathbf{k}}(\cdot,\eta) \le t < T^{\mathbf{k}\cdot 1}(\cdot,\eta), T^{\mathbf{k}}(\cdot,\eta) \le t < T^{\mathbf{k}\cdot 2}(\cdot,\eta)) \\ &= -2\sum_{n=1}^{\infty}\sum_{\mathbf{k} \in K^n} E_x\left[\int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)} \mu^{n-1} \binom{\eta(V_{s-})}{n-1} \mathbf{1}_{\{n-1 \le \eta(V_{s-})\}} P_{B_s}(t-s < S(\cdot,\eta_s))^2\right]\end{aligned}$$

$$= -2E_x\left[\int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)} \sum_{n=1}^{\eta(V_{s-})+1} (2\mu)^{n-1}\binom{\eta(V_{s-})}{n-1} E_{B_s}\left[e^{-\alpha\eta_s(V_{t-s})}\right]^2\right]$$

$$= -2E_x\left[\int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)} (1+2\mu)^{\eta(V_{s-})} E_{B_s}\left[e^{-\alpha\eta_s(V_{t-s})}\right]^2\right]$$

by the Markov property and the independence of $S^{\mathbf{k}\cdot 1}$ and $S^{\mathbf{k}\cdot 2}$.

We have

$$(\mathrm{II}) = \sum_{\substack{\mathbf{k},\tilde{\mathbf{k}}\in\mathbf{K}\setminus(K^0\cup K^1),\\ \mathbf{k}\neq\tilde{\mathbf{k}}}} \sum_{k,\tilde{k}\in\{1,2\}} \mathbf{1}_{\{T^{\mathbf{k}}\le t<T^{\mathbf{k}\cdot k}\}}\mathbf{1}_{\{T^{\tilde{\mathbf{k}}}\le t<T^{\tilde{\mathbf{k}}\cdot\tilde{k}}\}} = (\mathrm{II}') + (\mathrm{II}'')$$

for

$$(\mathrm{II}') := \sum_{m,\tilde{m}=1}^{\infty}\sum_{n=1}^{\infty} \sum_{\substack{\mathbf{b}\in\{1,2\}^m,\tilde{\mathbf{b}}\in\{1,2\}^{\tilde{m}},\\ b_1\neq\tilde{b}_1}} \sum_{\mathbf{a}\in K^n}\sum_{k,\tilde{k}\in\{1,2\}} \mathbf{1}_{\{T^{\mathbf{a}\cdot\mathbf{b}}\le t<T^{\mathbf{a}\cdot\mathbf{b}\cdot k}\}}\mathbf{1}_{\{T^{\mathbf{a}\cdot\tilde{\mathbf{b}}}\le t<T^{\mathbf{a}\cdot\tilde{\mathbf{b}}\cdot\tilde{k}}\}}$$

and

$$(\mathrm{II}'') := 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\mathbf{b}\in\{1,2\}^m}\sum_{\mathbf{a}\in K^n}\sum_{\substack{k,\tilde{k}\in\{1,2\},\\ k\neq b_1}} \mathbf{1}_{\{T^{\mathbf{a}}\le t<T^{\mathbf{a}\cdot k}\}}\mathbf{1}_{\{T^{\mathbf{a}\cdot\mathbf{b}}\le t<T^{\mathbf{a}\cdot\mathbf{b}\cdot\tilde{k}}\}}.$$

By the same calculation as that for $\mathbb{E}_x^{\eta}[(\mathrm{I})]$, it follows that

$$\mathbb{E}_x^{\eta}[(\mathrm{II}')] = \sum_{m,\tilde{m}=1}^{\infty}\sum_{n=1}^{\infty} \sum_{\substack{\mathbf{b}\in\{1,2\}^m,\tilde{\mathbf{b}}\in\{1,2\}^{\tilde{m}},\\ b_1\neq\tilde{b}_1}} \sum_{\mathbf{a}\in K^n}\sum_{k,\tilde{k}\in\{1,2\}} \mathbb{P}_x^{\eta}(T^{\mathbf{a}\cdot\mathbf{b}}\le t<T^{\mathbf{a}\cdot\mathbf{b}\cdot k}, T^{\mathbf{a}\cdot\tilde{\mathbf{b}}}\le t<T^{\mathbf{a}\cdot\tilde{\mathbf{b}}\cdot\tilde{k}})$$

$$= -2\sum_{m,\tilde{m}=1}^{\infty} E_x\Bigg[\int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)}(1+2\mu)^{\eta(V_{s-})}$$

$$\times E_{B_s}\Bigg[\int_{(0,t-s]} \mathrm{d}e^{-\alpha\eta_s(V_u)}\Bigg(2\cdot(2\mu)^{m-1}\binom{\eta_s(V_{u-})}{m-1}\mathbf{1}_{\{m-1\le\eta(V_{u-})\}}$$

$$-(2\mu)^m\binom{\eta_s(V_{u-})}{m}\mathbf{1}_{\{m\le\eta(V_{u-})\}}\Bigg)\Bigg]$$

$$\times E_{B_s}\Bigg[\int_{(0,t-s]} \mathrm{d}e^{-\alpha\eta_s(V_u)}\Bigg(2\cdot(2\mu)^{\tilde{m}-1}\binom{\eta_s(V_{u-})}{\tilde{m}-1}\mathbf{1}_{\{\tilde{m}-1\le\eta(V_{u-})\}}$$

$$-(2\mu)^{\tilde{m}}\binom{\eta_s(V_{u-})}{\tilde{m}}\mathbf{1}_{\{\tilde{m}\le\eta(V_{u-})\}}\Bigg)\Bigg]\Bigg]$$

and

$$\mathbb{E}_x^{\eta}[(\mathrm{II}'')] = 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\mathbf{b}\in\{1,2\}^m}\sum_{\mathbf{a}\in K^n}\sum_{\substack{k,\tilde{k}\in\{1,2\},\\ k\neq b_1}} \mathbb{P}_x^{\eta}(T^{\mathbf{a}}\le t<T^{\mathbf{a}\cdot k}, T^{\mathbf{a}\cdot\mathbf{b}}\le t<T^{\mathbf{a}\cdot\mathbf{b}\cdot\tilde{k}})$$

$$= 4\sum_{m=1}^{\infty} E_x\left[\int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)}(1+2\mu)^{\eta(V_{s-})} E_{B_s}\left[e^{-\alpha\eta_s(V_{t-s})}\right]\right.$$

$$\times E_{B_s}\left[\int_{(0,t-s]} \mathrm{d}e^{-\alpha\eta_s(V_u)}\left(2\cdot(2\mu)^{m-1}\binom{\eta_s(V_{u-})}{m-1}\mathbf{1}_{\{m-1\le\eta(V_{u-})\}}\right.\right.$$
$$\left.\left.-(2\mu)^m\binom{\eta_s(V_{u-})}{m}\mathbf{1}_{\{m\le\eta(V_{u-})\}}\right)\right]\Bigg].$$

Combining the calculations above with (3.2) and (3.4), we obtain (3.6). □

**Lemma 3.4** *For any* $f, g \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\begin{aligned}&\mathbb{E}_x[N_t(f)N_t(g)]\\&\quad= e^{\lambda t}E_x[f(B_t)g(B_t)]\\&\qquad+c\tilde{\mu}e^{2\lambda t}E_x\left[\int_0^t e^{-\lambda s}E_{B_s}\left[\exp\left(\lambda^2\int_0^{t-s}|U(B_u^1)\cap U(B_u^2)|\,\mathrm{d}u\right)f(B_{t-s}^1)g(B_{t-s}^2)\right]\mathrm{d}s\right].\end{aligned} \tag{3.7}$$

*In particular,*

$$\begin{aligned}\mathbb{E}_x[\overline{N}_t^2] &= e^{\lambda t}+c\tilde{\mu}e^{2\lambda t}\int_0^t e^{-\lambda s}E\left[\exp\left(\lambda^2\int_0^{2(t-s)}|U(B_u^1)\cap U(B_u^2)|\,\mathrm{d}u\right)\right]\mathrm{d}s\\&= e^{\lambda t}+c\tilde{\mu}e^{2\lambda t}\int_0^t e^{-\lambda s}E\left[\exp\left(\frac{\lambda^2}{2}\int_0^{2(t-s)}|U(0)\cap U(B_u)|\,\mathrm{d}u\right)\right]\mathrm{d}s.\end{aligned} \tag{3.8}$$

*Proof* We prove this lemma in a similar way to [9, Proposition 4.2.1(a)]. If $V_t^i$ denotes the tube $V_t$ with respect to $\{B_t^i\}_{t\ge 0}$ for each $i=1,2$, then we obtain

$$E_x\left[e^{\beta\eta(V_t)}f(B_t)\right]E_x\left[e^{\beta\eta(V_t)}g(B_t)\right]=E_x\left[e^{\beta(\eta(V_t^1)+\eta(V_t^2))}f(B_t^1)g(B_t^2)\right].$$

Define $V_t^1\Delta V_t^2=(V_t^1\setminus V_t^2)\cap(V_t^2\setminus V_t^1)$. Since $\eta(V_t^1\cap V_t^2)$ and $\eta(V_t^1\triangle V_t^2)$ are mutually independent and $\lambda(\beta)^2=\lambda(2\beta)-2\lambda(\beta)$ by definition, we have

$$\begin{aligned}Q\left[e^{\beta(\eta(V_t^1)+\eta(V_t^2))}\right]&=Q[\exp(2\beta\eta(V_t^1\cap V_t^2)+\beta\eta(V_t^1\triangle V_t^2))]\\&=\exp(\lambda(2\beta)|V_t^1\cap V_t^2|+\lambda(\beta)|V_t^1\triangle V_t^2|)\\&=\exp(\lambda(\beta)^2|V_t^1\cap V_t^2|)e^{2\lambda(\beta)t}\end{aligned}$$

from (3.5). Here we note

$$|V_t^1\cap V_t^2|=\int_0^t|U(B_s^1)\cap U(B_s^2)|\,\mathrm{d}s.$$

Set

$$U_{s,t}=\int_s^t|U(B_s^1)\cap U(B_s^2)|\,\mathrm{d}u\quad\text{and}\quad U_t=U_{0,t}\quad\text{for } 0\le s<t\le\infty.$$

We then get

$$\begin{aligned}&\mathbb{E}_x[N_t(f)N_t(g)]\\&\quad=e^{\lambda t}+c\tilde{\mu}e^{2\lambda t}E_x\left[Q\left[\int_{(0,t]}e^{-2\lambda s}e^{\beta\eta(V_{s-})}E_{B_s}\left[e^{\lambda^2U_{t-s}}f(B_{t-s}^1)g(B_{t-s}^2)\right]\mathrm{d}\eta(V_s)\right]\right]\end{aligned}$$

by the independence of the masses for disjoint sets of the Poisson random measure. Here we recall that, if a path $\omega \in \Omega$ is fixed, $\{\eta(V_t(\omega))\}_{t\geq 0}$ is a standard Poisson process on the half line. Since $e^{\beta\eta(V_{s-})}$ is predictable, the second term above is equal to

$$\begin{aligned} &c\tilde{\mu}e^{2\lambda t}E_x\left[Q\left[\int_0^t e^{-2\lambda s}e^{\beta\eta(V_{s-})}E_{B_s}\left[e^{\lambda^2 U_{t-s}}f(B^1_{t-s})g(B^2_{t-s})\right]\mathrm{d}s\right]\right] \\ &\quad = c\tilde{\mu}e^{2\lambda t}E_x\left[\int_0^t e^{-\lambda s}E_{B_s}\left[e^{\lambda^2 U_{t-s}}f(B^1_{t-s})g(B^2_{t-s})\right]\mathrm{d}s\right] \end{aligned}$$

(see [16, p. 472]), whence we have (3.7). Noting that (2.1) implies

$$\begin{aligned} \int_0^t |U(B^1_s)\cap U(B^2_s)|\,\mathrm{d}s &= \int_0^t |U(0)\cap U(B^2_s - B^1_s)|\,\mathrm{d}s \\ &\overset{\mathrm{d}}{=} \int_0^t |U(0)\cap U(B_{2s})|\,\mathrm{d}s = \frac{1}{2}\int_0^{2t} |U(0)\cap U(B_s)|\,\mathrm{d}s, \end{aligned}$$

we obtain (3.8). □

## 4 Proofs of Theorem 2.1 and Proposition 2.4

To establish Theorem 2.1, we prove

**Lemma 4.1** *Assume that* $P(U_\infty < \infty) = 1$. *Then*

$$\begin{aligned} &\lim_{t\to\infty} E_x\left[f(U_t)g\left(\frac{B^1_t}{\sqrt{t}}\right)\tilde{g}\left(\frac{B^2_t}{\sqrt{t}}\right)\right] \\ &\quad = E\left[f\left(\int_0^\infty |U(0)\cap U(B_{2s})|\,\mathrm{d}s\right)\right]\int_{\mathbb{R}^d} g(y)\rho(y)\,\mathrm{d}y\int_{\mathbb{R}^d}\tilde{g}(y)\rho(y)\,\mathrm{d}y \end{aligned}$$

*for any* $f, g, \tilde{g} \in C_b(\mathbb{R}^d)$.

*Proof* We prove this lemma by the same way as that in the proof of [7, Theorem 4.2]. By a standard approximation procedure, we may assume $f \in C_b(\mathbb{R}^d)$ and $g, \tilde{g} \in C_{b,u}(\mathbb{R}^d)$, where $C_{b,u}(\mathbb{R}^d)$ stands for the set of all bounded and uniformly continuous functions on $\mathbb{R}^d$. Then a direct calculation implies

$$\begin{aligned} &E_x\left[f(U_t)g\left(\frac{B^1_t}{\sqrt{t}}\right)\tilde{g}\left(\frac{B^2_t}{\sqrt{t}}\right)\right] \\ &\quad = E_x\left[f(U_s)g\left(\frac{B^1_t - B^1_s}{\sqrt{t}}\right)\tilde{g}\left(\frac{B^2_t - B^2_s}{\sqrt{t}}\right)\right] \\ &\qquad - E_x\left[f(U_s)g\left(\frac{B^1_t - B^1_s}{\sqrt{t}}\right)\tilde{g}\left(\frac{B^2_t - B^2_s}{\sqrt{t}}\right); U_{s,t} > 0\right] \\ &\qquad + E_x\left[f(U_s)\left(g\left(\frac{B^1_t}{\sqrt{t}}\right)\tilde{g}\left(\frac{B^2_t}{\sqrt{t}}\right) - g\left(\frac{B^1_t - B^1_s}{\sqrt{t}}\right)\tilde{g}\left(\frac{B^2_t - B^2_s}{\sqrt{t}}\right)\right); U_{s,t} = 0\right] \end{aligned}$$

$$+ E_x\left[f(U_t)g\left(\frac{B_t^1}{\sqrt{t}}\right)\tilde{g}\left(\frac{B_t^2}{\sqrt{t}}\right); U_{s,t} > 0\right]$$
$$=: (\mathrm{I}) - (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) \quad \text{for any } 0 \le s < t.$$

Since the two Brownian motions $\{B_t^1\}_{t\ge0}$ and $\{B_t^2\}_{t\ge0}$ are mutually independent with independent increments, we have

$$(\mathrm{I}) = E[f(U_s)]E\left[g\left(\frac{B_t^1 - B_s^1}{\sqrt{t}}\right)\right]E\left[\tilde{g}\left(\frac{B_t^2 - B_s^2}{\sqrt{t}}\right)\right]$$
$$\longrightarrow E\left[f\left(\int_0^\infty |U(0)\cap U(B_{2u})|\,\mathrm{d}u\right)\right]\int_{\mathbb{R}^d} g(y)\rho(y)\,\mathrm{d}y \int_{\mathbb{R}^d}\tilde{g}(y)\rho(y)\,\mathrm{d}y$$
as $t \to \infty$ and then $s \to \infty$.

By assumption, we get

$$|(\mathrm{II})| \le \|f\|_\infty\|g\|_\infty\|\tilde{g}\|_\infty P(U_{s,t} > 0) \le \|f\|_\infty\|g\|_\infty\|\tilde{g}\|_\infty P(U_{s,\infty} > 0) \longrightarrow 0 \quad \text{as } s \to \infty.$$

By the same way, $\lim_{s\to\infty}(\lim_{t\to\infty}(\mathrm{IV})) = 0$ follows. Since $g$ and $\tilde{g}$ belong to $C_{b,u}(\mathbb{R}^d)$, we obtain $\lim_{t\to\infty}(\mathrm{III}) = 0$. These complete the proof. □

*Proof of Theorem 2.1* We first prove the implication (i) ⇔ (ii). From (3.8), we have

$$\sup_{t\ge0}\mathbb{E}[\overline{M}_t^2] = \lim_{t\to\infty}\mathbb{E}[\overline{M}_t^2] = \frac{c}{m^{(1)}-1}E\left[\exp\left(\frac{\lambda^2}{2}\int_0^\infty |U(0)\cap U(B_s)|\,\mathrm{d}s\right)\right],$$

which shows the implication (i) ⇔ (ii).

We next prove the implication (ii) ⇒ (iii). Set

$$L_t(f) = \int_{\mathbb{R}^d} f\left(\frac{x}{\sqrt{t}}\right)M_t(\mathrm{d}x).$$

Then by (3.7), we get

$$\mathbb{E}[L_t(f)^2] = e^{-\lambda t}E\left[f\left(\frac{B_t}{\sqrt{t}}\right)^2\right] + c\tilde{\mu}E\left[\int_0^t e^{-\lambda s}E_{B_s}\left[e^{\lambda^2 U_{t-s}}f\left(\frac{B_{t-s}^1}{\sqrt{t}}\right)f\left(\frac{B_{t-s}^2}{\sqrt{t}}\right)\right]\mathrm{d}s\right].$$

By Lemma 4.1 applied to the second term of the right hand side of the equality above, we have

$$\lim_{t\to\infty}\mathbb{E}[L_t^2] = \frac{c}{m^{(1)}-1}E\left[\exp\left(\frac{\lambda^2}{2}\int_0^\infty |U(0)\cap U(B_s)|\,\mathrm{d}s\right)\right]\left(\int_{\mathbb{R}^d} f(x)\rho(x)\,\mathrm{d}x\right)^2$$
$$= \mathbb{E}[\overline{M}_\infty^2]\left(\int_{\mathbb{R}^d} f(x)\rho(x)\,\mathrm{d}x\right)^2, \tag{4.1}$$

which shows the implication (ii) ⇒ (iii). The implication (iii) ⇒ (ii) follows by taking $f \equiv 1$ in (4.1). □

*Proof of Proposition 2.4* Since

$$R_t = \frac{1}{\overline{M}_t^2}\int_{\mathbb{R}^d} M_t(U(x))^2\,\mathrm{d}x,$$

it is enough to show

$$\mathbb{E}\Big[\int_{\mathbb{R}^d} M_t(U(x))^2 \,\mathrm{d}x\Big] = O(t^{-d/2}). \tag{4.2}$$

It follows from (3.7) that

$$\begin{aligned}\mathbb{E}\Big[\int_{\mathbb{R}^d} M_t(U(x))^2 \,\mathrm{d}x\Big] &= \int_{\mathbb{R}^d} \mathbb{E}[M_t(U(x))^2] \,\mathrm{d}x \\ &= e^{-\lambda t}\int_{\mathbb{R}^d} P(B_t \in U(x)) \,\mathrm{d}x \\ &\quad + c\tilde{\mu} E\Big[\int_0^t \mathrm{d}s\, e^{-\lambda s}\int_{\mathbb{R}^d} E_{B_s}\big[e^{\lambda^2 U_{t-s}}; B^1_{t-s}, B^2_{t-s} \in U(x)\big] \,\mathrm{d}x\Big].\end{aligned} \tag{4.3}$$

By the equality

$$E_{B_s}\big[e^{\lambda^2 U_{t-s}}; B^1_{t-s}, B^2_{t-s} \in U(x)\big] = E_{B_s}\big[e^{\lambda^2 U_{t-s}}; x - B^1_{t-s} \in U(0) \cap U(B^2_{t-s} - B^1_{t-s})\big],$$

we obtain

$$\int_{\mathbb{R}^d} E_{B_s}\big[e^{\lambda^2 U_{t-s}}; B^1_{t-s}, B^2_{t-s} \in U(x)\big]\mathrm{d}x = \int_{\mathbb{R}^d} E_{B_s}\big[e^{\lambda^2 U_{t-s}}; x \in U(0) \cap U(B^2_{t-s} - B^1_{t-s})\big] \,\mathrm{d}x.$$

In addition, the last term above is equal to

$$\begin{aligned}&\int_{U(0)} E_{B_s}\big[e^{\lambda^2 U_{t-s}}; B^2_{t-s} - B^1_{t-s} \in U(x)\big] \,\mathrm{d}x \\ &\quad = \int_{U(0)} E\Big[\exp\Big(\frac{\lambda^2}{2}\int_0^{2(t-s)} |U(0) \cap U(B_u)| \,\mathrm{d}u\Big); B_{2(t-s)} \in U(x)\Big] \,\mathrm{d}x\end{aligned}$$

by (2.1). Furthermore, by [8, Lemma 3.1.4], the last term above is not greater than

$$\frac{C_1}{(t-s)^{d/2}}\int_{U(0)}\Big(\int_{U(x)} \mathrm{d}y\Big) \,\mathrm{d}x = \frac{C_1}{(t-s)^{d/2}} \quad \text{for any } t > s > 0$$

with some positive constant $C_1 > 0$. Therefore, the right hand side of (4.3) is not greater than

$$e^{-\lambda t} + \frac{C_1 c\tilde{\mu}}{t^{d/2}}\int_0^{t/2} e^{-\lambda s} \,\mathrm{d}s + c\tilde{\mu}\Big(\int_{t/2}^t e^{-\lambda s} \,\mathrm{d}s\Big) E\Big[\exp\Big(\frac{\lambda^2}{2}\int_0^\infty |U(0) \cap U(B_s)| \,\mathrm{d}s\Big)\Big],$$

which leads to (4.2). □

*Remark 4.2* (Extinction) Let $(\{Y_t\}_{t\geq 0}, \mathbf{P})$ be a continuous time branching process with $\tilde{\mu} = 1 - e^{-\alpha}$ and $\{p_n\}_{n=0}^\infty$ as branching rate and as offspring distribution, respectively. We then have the inequality

$$\mathbf{P}\Big(\lim_{t\to\infty} Y_t = 0\Big) \leq \mathbb{P}\Big(\lim_{t\to\infty} \overline{N}_t = 0\Big). \tag{4.4}$$

We prove this inequality by the same way as that for the discrete time-space case (see [4, Theorem 4]). It is known by [1, p. 108, Theorem 1] that the extinction probability $\mathbf{P}(\lim_{t\to\infty} Y_t = 0)$ is the smallest root of the equation

$$u = \sum_{n=0}^{\infty} p_n u^n, \quad 0 \le u \le 1.$$

On the other hand, we get

$$\mathbb{P}(\overline{N}_t = 0) = \tilde{\mu}\mathbb{E}\left[\int_{(0,t]} \mathrm{d}e^{-\alpha\eta(V_s)} \sum_{n=0}^{\infty} p_n \mathbb{P}_{B_s}^{\eta_s}(\overline{N}_{t-s} = 0)^n\right].$$

Then, by letting $t$ go infinity and then applying the monotone convergence theorem for this equality, we have

$$\mathbb{P}\Big(\lim_{t\to\infty} \overline{N}_t = 0\Big) = \tilde{\mu}\mathbb{E}\left[\int_{(0,\infty)} \mathrm{d}e^{-\alpha\eta(V_s)} \sum_{n=0}^{\infty} p_n \mathbb{P}_{B_s}^{\eta_s}\Big(\lim_{t\to\infty} \overline{N}_t = 0\Big)^n\right].$$

Since $\eta(V_s)$ and $\mathbb{P}_{B_s(\omega)}^{\eta_s}(\overline{M}_\infty = 0)$ are independent for each $\omega \in \Omega$, the right hand side above is not less than

$$\tilde{\mu}\mathbb{E}\left[\int_{(0,\infty)} \mathrm{d}e^{-\alpha\eta(V_s)} \sum_{n=0}^{\infty} p_n Q^{\mathcal{G}_s}\Big[\mathbb{P}_{B_s}^{\eta_s}\Big(\lim_{t\to\infty} \overline{N}_t = 0\Big)\Big]^n\right] = \sum_{n=0}^{\infty} p_n \mathbb{P}\Big(\lim_{t\to\infty} \overline{N}_t = 0\Big)^n$$

by Schwarz's inequality, where $Q^{\mathcal{G}_s}$ is the conditional expectation given $\mathcal{G}_s$. Namely, we get

$$\mathbb{P}\Big(\lim_{t\to\infty} \overline{N}_t = 0\Big) \ge \sum_{n=0}^{\infty} p_n \mathbb{P}\Big(\lim_{t\to\infty} \overline{N}_t = 0\Big)^n,$$

whence (4.4) holds by the characterization of $\mathbf{P}(\lim_{t\to\infty} Y_t = 0)$ as we mentioned above. In particular, we have $\mathbb{P}(\lim_{t\to\infty} \overline{N}_t = 0) = 1$ for $m^{(1)} < 1$ because $\mathbf{P}(\lim_{t\to\infty} Y_t = 0) = 1$ holds.

## 5 Connection with Brownian Directed Polymers in Random Environment

In this section, we confirm a connection between the model of branching Brownian motions in random environment and the model of Brownian directed polymers in random environment introduced by Comets and Yoshida [9]. Let $\mu_t^x$ be a probability measure on $(\Omega, \mathcal{F})$, the so-called polymer measure, defined by

$$\mu_t^x(\mathrm{d}\omega) = \frac{e^{\beta\eta(V_t)}}{Z_t^x} P_x(\mathrm{d}\omega), \quad \eta \in \mathcal{M}.$$

Here $\beta \in \mathbb{R}$ is a parameter and

$$Z_t^x = E_x\big[e^{\beta\eta(V_t)}\big],$$

which is called the partition function. The size of $\eta(V_t)$ is considered as the total number of impurities governed by $\eta$ in the tube $V_t$, and thus the polymer measure is nothing but the law of the Brownian motion in environment $\eta$.

Let

$$W_t = e^{-\lambda t} Z_t$$

for $\lambda = \lambda(\beta) = e^{\beta} - 1$ as we defined in (2.2). Then $W_t$ is called the normalized partition function because $Q[W_t] = 1$ holds. In addition, since the process $\{\eta(V_t(\omega))\}_{t \geq 0}$ has independent Poisson increments for each $\omega \in \Omega$, $W_t$ is a mean-one, right continuous and left limited, positive martingale on $(\mathcal{M}, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0}, Q)$. In particular, the limit $W_\infty := \lim_{t\to\infty} W_t$ exists $Q$-a.s. Since $e^{\beta\eta(V_t)} > 0$ holds for all $t \geq 0$, the event $\{W_\infty = 0\}$ is measurable with respect to the tail $\sigma$-field

$$\bigcap_{t \geq 1} \sigma(\eta|_{[t,\infty)\times\mathbb{R}^d}).$$

Furthermore, Kolmogorov's 0–1 law implies $Q(W_\infty > 0) = 1$ or $Q(W_\infty = 0) = 1$. The situation $Q(W_\infty > 0) = 1$ is called the weak disorder and another situation $Q(W_\infty = 0) = 1$ the strong disorder.

Since (3.2) yields

$$\mathbb{E}^\eta\big[N_t(A)\big] = E\big[e^{\beta\eta(V_t)}; B_t \in A\big] \quad \text{and} \quad \mathbb{E}^\eta\big[\overline{N}_t\big] = Z_t \tag{5.1}$$

for any $\eta \in \mathcal{M}$, we obtain

$$\mathbb{E}^\eta\big[M_t(A)\big] = e^{-\lambda t} E\big[e^{\beta\eta(V_t)}; B_t \in A\big] \quad \text{and} \quad \mathbb{E}^\eta\big[\overline{M}_t\big] = W_t, \tag{5.2}$$

and thus

$$\mu_t(B_t \in A) = \frac{\mathbb{E}^\eta[N_t(A)]}{\mathbb{E}^\eta[\overline{N}_t]} = \frac{\mathbb{E}^\eta[M_t(A)]}{\mathbb{E}^\eta[\overline{M}_t]}.$$

Theorem 2.1 then leads to the central limit theorem which is weaker than that proved in [8, Theorem 2.1.1].

**Corollary 5.1** *Assume $d \geq 3$ and $\beta > 0$. If one of the conditions in Theorem 2.1 holds, then*

$$\lim_{t\to\infty} \mu_t\left[f\left(\frac{B_t}{\sqrt{t}}\right)\right] = \int_{\mathbb{R}^d} f(x)\rho(x)\,\mathrm{d}x \quad \textit{in } Q\textit{-probability}$$

*for any $f \in C_b(\mathbb{R}^d)$.*

Comets and Yoshida [9, Theorem 2.1.1] showed the existence of the phase transition for Brownian directed polymers in random environment in terms of the weak disorder and the strong disorder, or the growth rate of the partition function. Using the relation (5.2), we can derive the existence of the phase transition also for branching Brownian motions in random environment in terms of the population growth rate. To do this, we prove

**Proposition 5.2**

(i) *The inequality*

$$\mathbb{E}\big[\overline{M}_\infty\big] \leq Q[W_\infty]$$

*holds. In particular, $Q(W_\infty = 0) = 1$ implies $\mathbb{P}(\overline{M}_\infty = 0) = 1$.*

(ii) *The converse of the inequality above does not hold for $d \geq 3$.*

*Proof* (i) It follows from Fatou's lemma that

$$\mathbb{E}\big[\overline{M}_\infty\big] \le Q\Big[\lim_{t\to\infty} \mathbb{E}^\eta\big[\overline{M}_t\big]\Big] = Q\Big[\lim_{t\to\infty} W_t\Big] = Q[W_\infty].$$

(ii) Assume $d \ge 3$ and $m^{(1)} < 1$. Then it follows that $\mathbb{P}(\lim_{t\to\infty} \overline{N}_t = 0) = 1$ as we mentioned in Remark 4.2, and thus $\mathbb{P}(\overline{M}_\infty = 0) = 1$. However, since $\beta = \log\{1 + (1 - e^{-\alpha})(m^{(1)} - 1)\}$ is negative, we have $Q(W_\infty > 0) = 1$ by [9, Theorem 2.1.1(c)]. □

If $d = 1$ or 2, then the correlation among particles is strong as we mentioned in Remark 2.2. Even if $d \ge 3$, the situation is the same as the low dimensional case when the parameter $\beta$ is large enough. In fact, it follows from Bertin [2, 3] and Comets and Yoshida [9, Theorems 2.1.1 and 2.2.2] that

**Corollary 5.3** *For $d = 1$ or 2, $\mathbb{P}(\overline{M}_\infty = 0) = 1$ holds for any $\beta > 0$. On the other hand, for $d \ge 3$, there exists a positive constant $\beta_0(d) > 0$ such that $\mathbb{P}(\overline{M}_\infty = 0) = 1$ holds for any $\beta \in (\beta_0(d), \infty)$. Moreover, for any dimension $d$, there exists a non-negative constant $\beta_1(d) \ge 0$ such that for any $\beta \in (\beta_1(d), \infty)$,*

$$\limsup_{t\to\infty} \frac{\log \overline{M}_t}{t} < -c(\beta) \quad \mathbb{P}\text{-}a.s.$$

*holds with a non-random constant $c(\beta) > 0$. In particular, $\beta_1(1) = \beta_1(2) = 0$ and $\beta_1(d) > 0$ if $d \ge 3$.*

Corollary 5.3 says that, if the randomness of the environment is strong enough, the growth rate of the population size is strictly less than its expectation almost surely. This result contrasts with the non-random environment case and the weak random environment case as we discussed before.

*Remark 5.4* It is recently proved in [19] that, if $\mathbb{P}(\overline{M}_\infty = 0) = 1$, there exist non-random positive constants $c_1, c_2 > 0$ such that

$$-c_1 \log \overline{M}_t \le \int_0^t R_s \, \mathrm{d}s \le -c_2 \log \overline{M}_t \quad \text{for any } t \ge T \ \mathbb{P}\text{-a.s.}$$

holds with some random positive constant $T$. Combining this with Corollary 5.3, we have for any $\beta > \beta_1(d)$,

$$\limsup_{t\to\infty} \overline{\rho}_t \ge \limsup_{t\to\infty} R_t \ge c'(\beta) \quad \mathbb{P}\text{-a.s.}$$

with some non-random positive constant $c'(\beta) \in (0, 1)$. Namely, if the randomness of the environment dominates that of the Brownian motion, particles gather together at small sets in contrast with the diffusive behavior as we proved in Corollary 2.1 above. This result is an extension of that obtained by Hu and Yoshida [10] for branching random walks in random environment.

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